A STUDY AND THE USE OF LAGRANGE MULTIPLIER IN CALCULUS OF VARIATION

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Declaration

I Abdulyekeen Khadijat Oluwakemi, hereby declare that this research project titled has been carried out by me under the supervisions of Dr O.E Abolarin and Mrs O.R Ajewole. All sources of information is specifically acknowledged by means of reference.

Certification

This is to certify that this research project titled A STUDY AND THE USE OF LAGRANGE MULTIPLIER IN CALCULUS OF VARIATION by ABDULYEKEEN KHADIJAT OLUWAKEMI meeting the requirements governing the award of Degree of Bachelor of Science in Mathematics and approved for its contribution to knowledge and literary representation.

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Dedication

This project work is dedicated to the glory of God for His mercy and grace upon my life, and to my parents, guardian, sister, brothers and loved ones.

Acknowledgement

My sincere gratitude goes to the Almighty God for His sufficient grace upon my life, especially for given me a sound health, knowledge and understanding throughout my study. My gratitude also goes to my supervisor Dr O.E Abolarin and Mrs O.R Ajewole for taking time to look into this project work and making all the necessary corrections, and to all my lecturers in the department for the knowledge impacted into me throughout my course of study, more of God's wisdom. I must not fail to acknowledge the effort of my wonderful parent Mr.and Mrs.Abdulyekeen Jimoh, big mummy and my beloved sister and brothers for their financial support and encouragement. Furthermore I would like to recognize the contribution of my friends, well wishers and colleagues who went through the same program with me, God bless you all (Amen).

Abstract

This project work examines the use of Lagrange multipliers to calculus of variation (isoperimetric problem). Basic definition of terms were given, necessary and sufficient condition for a function to be maxima or minima, how to identify Lagrange multipliers.in any given problem and general useage of largange multipliers, Lagrange multiplier in unconstraint and constraint problems, theorems and proof related to Lagrange multipliers. Literature review, Euler's Multiplier rule and isoperimetric problem, proof's motivated by Euler and Lagrange, the power system economic operation.
Methods of solving Lagrange function, i also included derivation of Euler-Lagrange equation and other form's of Euler equation, extremal,calculus of variation, isoperimetric problems and method for solving extrema of a given function (minimum and maximum) were examined. Numerical examples were provided.

Contents

Chapter 1

Introduction

Lagrange multipliers are useful techniques in multivariate calculus, one of the most common problems in calculus is that of finding minima and maxima (in general extrema) of a function, Lagrange multiplier is a powerful method for solving these kind of problems. Mathematical optimization, according to Joseph Louis Lagrange, is a strategy for finding the maxima and minima of a function subject to equality and inequality constraints. For instance $f(x_0, x_1, x_2, ..., x_n)$ subject to a constraint $g(x_0, x_1, x_2, ..., x_n) = 0$, where f and g are functions with continuous first partial derivatives. We introduce a new variable λ called Lagrange multiplier and study the Lagrangian function as defined below.

$$\Delta(x, y, \lambda) = f(x_0, x_1, x_2, ..., x_n) + \lambda g(x_0, x_1, x_2, ..., x_n)$$

If (x_0, y_0) is a maximum of $f(x, y)$ for the constrained problem, then there exist λ_0 Such that (x_0, y_0, λ_0) are the stationary points for the Lagrange function (stationary points are those points where the partial derivatives are zero).

However, not all the stationary points yield a solution of the original problem, thus the method of Lagrange multipliers yields a necessary condition and Sufficient conditions for constrained problems.

1.1 Aims And Objectives Of The Study

The aims and objectives of the study are to know the method of Lagrange multiplier, when to use and apply it to different fields specifically in calculus of variation (isoperimetric problem) finding the maximum and minimum of a multivariate function under some specific conditions known as constraints

1.2 Relevance Of The Study

Lagrange multiplier is widely used to solve extrema value problems in field of science, social science and engineering. It also helps in getting the stationary point or critical point, minima and maxima of a given function, it plays an important role in our everyday life activities e.g.like finding the shortest distance of a location.

1.3 Range Of The Study

The study will highlight some uses and applications of Lagrange multipliers most especially in calculus of variation(isoperimetric problems).

1.4 Definition Of Terms

Before we can see why the method of Lagrange multipliers work the way it works, there are some terms we need to understand for the unconstrained optimization problems.

- ### Gradient Of A Function Of Two Variables

  A gradient is just a vector that collects all functions of partial first derivatives in one place, in mathematics a gradient is a generalization of a function in one dimension to a function in several dimensions.

  Definition

  Let Z = $f(x, y)$ be a function such that f_y and f_x exist. Then the gradient of f denoted by $\nabla f(x, y)$ are vectors of the two variables.

  $$\nabla f(x, y) = f_x(x, y) + f_y(x, y)$$

  ∇f is read as "del$f$" another notation for the gradient is grad $f(x, y)$

- ### Stationary Point

  A function,either single variable or multivariate is said to be at stationary point or (critical point) if given a function $y = f(x)$ such that

  $$\frac{df}{dx} = f_x = 0$$

In the case of a function $y = f(x)$ of a single variable a stationary point can be any of the following three

1. maximum point
2. minimum point
3. inflection point

But for a function of two variables $y = f(x, y)$, the stationary point can be

1. maximum point
2. minimum point
3. saddle point

Maxima And Minima Of A Function

- A function $f(x, y)$ is said to be maximum at a point (x_0, y_0), if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in the neighborhood of (x_0, y_0).
  A function $f(x, y)$ is said to be minimum at a point (x_0, y_0), if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in the neighborhood of (x_0, y_0).

- ## Local Maximum And Local Minimum

  Function can have hills and valleys i.e. places where they reach minimum or maximum values. It may be the minimum or maximum for the whole function but locally (maximum or minimum)as shown in figure(1) of the appendix

Definition

Local maximum is where the height of the function at a is greater than (or equal to) the height anywhere else in that interval i.e.$f(a) \geq f(x)$ for all x in the interval.In other word, there is no height greater than $f(a)$, as shown in figure(2) in the appendix

Definition

Local minimum: This is the point where the height of the function at a is lesser than (or equal to) the height anywhere else in the interval i.e. $f(a) \leq f(x)$ for all x in the interval, as shown in figure(2)in the appendix

- ## Absolute Maximum And Minimum

An absolute maximum of a function f on a set S occurs at x_0 in S if $f(x) \leq f(x_0)$ for all x in S.
An absolute minimum of a function f on a set S occurs at x_0 in S if $f(x) \geq f(x_0)$ for all x in S.
Absolute maximum and minimum is also know as Global maximum and minimum as shown in fig(3) in the appendix

- ## Constraint

A constraint is a relationship that satisfies feasible values for a structural variable

- ## Functional

A real valued function f whose domain is the set of real function $y(x)$ is known as a functional (or functional of single independent variables). Thus the domain definition of a functional is a set of admissible function

Definition

A functional $I[y(x)]$ attains a maximum on a curve $y = y_0(x)$ if the value of I on any closed curve to $y = y_0(x)$ does not exceed $I[y_0(x)]$. This means that $\Delta I = I[y(x)]$ - $I[y_0(x)] \leq 0$, i.e. if $\Delta I \leq 0$ and $\Delta I = 0$ on $y = y_0(x)$ then a maximum is attained on $y = y_0(x)$.
A functional $I[y(x)]$ attains a minimum on a curve $y = y_0(x)$ if the value of I on any close curve to $y = y_0(x)$ exceed $I[y_0(x)]$ this means that $\Delta I = I[y(x)]$ - $I[y_0(x)] \geq 0$,i.e. if $\Delta I \geq 0$ and $\Delta I = 0$ on y then a minimum is attained on $y = y_0(x)$.

1.5 Necessary Condition For A Function To Be Maxima And Minima

The necessary condition for a function of two variables $f(x, y)$ to have a minima or maxima at a point (x_0, y_0) is when

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \tag{1.1}$$

this is called the stationary point. To find the minimum or maximum of a function we first locate the stationary point, then examine each stationary point to determine if it is maximum or minimum and to determine if a point is maximum or minimum, we may consider the value of the function in the neighborhood of the point as well as the value of its first and second partial derivatives.

1.6 Sufficient Condition For A Function To Be Maxima And Minima

let $Z = f(x, y)$ be continuous function in first and second partial derivatives in the neighborhood of point (x_0, y_0) then if

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

$$(\frac{\partial^2 f}{\partial x \partial y})^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} < 0 \tag{1.2}$$

There exists a saddle point, meaning it is a point where function is neither maximum nor minimum. Then there exist a maximum at (x_0, y_0) if

$$\frac{\partial^2 f}{\partial x \partial y} < 0 \tag{1.3}$$

There exist a minimum at (x_0, y_0) if

$$\frac{\partial^2 f}{\partial x \partial y} > 0 \tag{1.4}$$

Note: All these conditions are applicable to one, two or n variables

Example 1.5.1

Consider the function $f(x) = 7x^2 - 3x + 5$

In solving this kind of problem we need to find the first derivative and our critical point

$$f'(x) = 14x - 3$$

we then have $14x - 3 = 0$ and our critical point is now $\frac{3}{14}$ and $f''(x) = 14 > 0$, which is minimum since $\frac{d^2f}{dx^2} > 0$
Then $f(x))$ is minimum at point $\frac{3}{14}$

Example 1.5.2

Consider the function $f(x) = x^3 - 2x^2 + x + 1$

solution

$f'(x) = 3x^2 - 4x + 1 = 0$
solving $3x^2 - 4x + 1 = 0$ we have our critical number to be 1 and $\frac{1}{3}$

$$f''(x) = 6x - 4$$

$$f''(1) = 6 - 4 = 2 > 0$$

$$f''(\frac{1}{3}) = 2 - 4 = -2 < 0$$

so f has a relative minimum at 1 and relative maximum at $\frac{1}{3}$

1.7 Constraint

A constraint in the mathematical sense is a limitation usually imposed upon either the domain of a function or range of a function. For example, one might say, find the solution to the equation $x^2 = 9$, subject to $x \geq 0$, $x \geq 0$ is a constraint because it limits the answer to a positive number only. Without that constraint, one would have to include -3 as a solution but since $x \geq 0$, it means $x = 3$ is our only answer.

Example

$$y' = 2x$$

$$\frac{dy}{dx} = 2x$$

$$\int dy = \int 2x dx$$

$$y = x^2 + C$$

Given a constraint, with these we can now get our particular solution. Since $y(1)$=7 it means $x = 1$ and $y = 7$ from our constraint

$$7 = 1 + c$$

$$C = 6$$

$$y = x^2 + 6$$

Integral Constraints

These are constraints that are in integral form, isoperimetric problems in calculus of variation are examples of such problems where an integral is to be optimized, subject to a constraint which is another integral having a specified value. This name came from the famous problem of Dido of finding the closed curve of a given perimeter for which the area is a maximum or minimum for the Euler equation the problem can be stated as

$$Optimize I y(x) = \int_{x_0}^{x} f(x, y, y') dx$$

Subject to

$$J = \int_{x_0}^{x} G(x, y, y')dx$$

Where J is the constraint

Equality Constraints

$$Minf(x_1, ..., x_n)$$

Subject to

$$G(x_1, ..., x_n) = 0$$

$$where[G_1(X_1, ..., X_n)...G_n(x_1, ..., x_n)]$$

The Lagrange function f is constructed as

$$f(x, \lambda) = f(x) - \lambda G(x)$$

Where $x = (x_1, ...x_n)$ the variable $\lambda = [\lambda_1, ..., \lambda_n]$, where $\lambda_1, ..., \lambda_n$ are called Lagrange multipliers. The extrema points of the f and the Lagrange multipliers satisfy:

$$\Delta f = 0$$

$$\frac{df}{dx_i} - \sum_{m=0}^{k} \lambda_m \frac{\partial G_m}{x_i} = 0 i = 1, 2, ..., n$$

$$G(x_1, ..., x_n) = 0$$

Lagrange multipliers method defines the necessary conditions for the constrained nonlinear optimization problems.

Inequality Constraints

The Lagrange multipliers method also covers the case of inequality constraints,

$$Minf(x_1, ..., x_n)$$

Subject to:

$$G(x_1, ..., x_n) = 0$$

$$H(x_1, ..., x_n) \geq 0$$

In the feasible region $H(x_1, ..., x_n) = 0$ or $H(x_1, ..., x_n) < 0$ where $H_i = 0$, H is said to be active, otherwise H_i is inactive. The augmented Lagrange function is now

$$\text{Subject to:}G(x_1, ..., x_n) = 0$$

Then we have the lagrangian function to be

$$f(x, \lambda, \mu) = f(x) - \lambda G(x) - \mu H(x)$$

$$[H_1(X_1, ..., X_n)...H_n(x_1, ..., x_n)]$$

$$\mu = (\mu_1, ..., \mu_n)$$

When H_i is inactive, we can simply remove the constraint by setting μ_i = 0. If $\Delta f < 0$, it points to the descending direction of f and when H_i is active, this direction points out of the feasible region and towards the forbidden side, which means $\Delta H_i > 0$. This is not the solution direction. We can enforce $\mu_i \leq 0$ to keep the seeking direction still in the feasible region. When extended to cover the inequality constraints, the rule for the Lagrange multipliers method can be generalized as

$$\Delta f(x) - \sum_{t=1}^{k} \lambda_t \Delta G_t(x) - \sum_{j=1}^{m} \Delta H_j(x) = 0$$

$$\mu_i, H_i \leq 0 i = (1, 2, 3, ...m)$$

$$\mu_i, H_i = 0 i = (1, 2, 3, ...m)$$

$$.G(x) = 0$$

In summary, for inequality constraints, we add them to the Lagrange function just as if they are equality constraints, except that we require that $\mu_i \leq 0$ and when $H_i \neq 0$, $\mu_i = 0$.

1.8 Some Useful Theorems Used In Lagrange Multipliers

Theorem

Let f and g have first partial derivatives such that f has extremum at a point(x_0, y_0) on the smooth constraint curve $g(x, y)$ = C. If $\nabla g(x_0, y_0) \neq 0$ then there is a real numberλ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Theorem

For a constrained system local maxima and minima (collectively extrema) occur at the critical points. Suppose M is the maximum (or minimum) value of $f(x,y)$, subject to the constraint $g(x,y) =$ c. The Lagrange multiplier λ is the rate of change of M with respect to C. That is,

$$\lambda = \frac{\delta M}{\delta C}$$

Theorem

If a function $I[y(x)]$ attains a maximum or minimum on y $=y_0(x)$ where the domain of definition belongs to certain class, then at y $= y_0(x)$ then $\delta I = 0$

Proof

For fixed $y_0(x)$ and δy, $I[y_0(x) + a\delta y] = \psi(a)$, where ψ is a function of a and this reaches a maximum or minimum at $a = 0$ thus, $\varphi'(0) = 0$ leading to

$$\frac{\partial}{\partial a} I[Y_0(X) + a\delta y]|_|0 = 0 i.e \delta I = 0$$

If a function $I[y(x)]$ attains a maximum or minimum on the curve $y = y_0(x)$ with respect to curve $y = y(x)$, such that $[y(x) - y_0(x)]$ is small then the maximum or minimum is said to be strong.
On the other hand, if $I[y(x)]$ attains a maximum or minimum on the curve y $= y_0(x)$ with respect to curve $y = y(x)$ such that $[y(x) - y_0(x)]$ and $[y'(x) - y_0'(x)]$ are both small then the maximum or minimum is said to be weak.

Chapter 2

literature Review

This is the historical background of how isoperimetric problems came about in the early Greeks. Isoperimetry (the study of geometric figures of equal perimeters) was a topic well embraced by the ancient Greeks. Yet the Greeks did not have a clear understanding of the relationship between perimeter and area. Proclus claims that this lack of understanding led to cheating in land dealings.
Moreover, the theorem that all triangles formed on the same base and always between the same two parallel lines are equal in area was considered paradoxical by the Greeks since the perimeter could be made as large as possible. In spite of this the Greeks were outstanding and proved that the equilateral triangle solved the isoperimetric problem for the triangle and that the Square solve the isoperimetric problem for the rectangle. The origin of the isoperimetric problem should be attributed to the early Greeks because it is not known who among them was the first to state the problem, state the solution, or attempt a solution, Some historians claim that Pythagorus (580 BC - 500 BC) knew the maximum principle of the circle. However, Porter claims that Pythagorus knowledge was no deeper than believing that of all plane figures the circle is the most beautiful. Porter dismisses this statement, with all the augment they still have some difficulty completely divorcing pathagorus statement from the iso-area problem.

Mathematical historians tend to agree that Archimedes (287 BC - 212 BC) was well aware of the isoperimetric problem and in its solution. However there is no agreement as to whether or not he attempted the proof. Zenodorus (200 BC - 140 BC) authored a book entitled "On Isoperimetric Figures". This book was unfortunately lost, but the work has been partially preserved by Theon (335 AD - 405 AD) and Pappus (290 AD - 350 AD), his preserved work includes the following two theorems.

Theorem 2.0.1

Among all polygons of equal number of sides and equal perimeters, the regular polygon encloses the greatest area.

Theorem 2.0.2

The circle has greater area than any regular polygon of equal perimeter.

"Porter" notes that Zenodorus assumed existence of a solution in his proof of Theorem 2.0.1 and this gap in his proof was corrected by Weierstrass two thousand years later. Historians and mathematicians alike credit Zenodorus with the first attempt to prove that the circle solves the isoperimetric problem, claiming that the proof either contained a flaw or was incomplete. However, that Zenodorus attempted a proof cannot be validated from looking at his work preserved by Theon or Pappus. Hence, it could just be that what some are referring to as an incomplete proof is merely the proof of Theorem 2.0.2 stated above. However, it was believed that it is more likely that Zenodorus merely stated that the circle solves the isoperimetric problem in two dimensions and the sphere solves the isoperimetric problem in three dimensions.

2.1 Euler's Multiplier Rule and the Isoperimetric Problem.

First Euler in 1744 and later Lagrange in 1759, in a different manner demonstrated that a solution of problem below must satisfy the so-called Euler-Lagrange equation

Extremize f(y)= $\int_a^b f(x, y, y')$

subject to G(Y)= $\int_a^b g(x, y, y')$ (2.1)

Equation(2.1) must satisfy the Euler's equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} \tag{2.2}$$

Solutions of the Euler-Lagrange equation are then called extremals of problem of (2.1). Euler explains how to handle problems where, in addition to the boundary conditions, the solution must satisfy a subsidiary condition (constraint). He considered the problem

Extremize f(y)= $\int_a^b f(x, y, y')$

Subject to G(Y)= $\int_a^b g(x, y, y') = l$

$$y(a) = \alpha \text{and} y(b) = \beta \quad (2.3)$$

standard assumptions made by Euler and Lagrange that (equation 2.1) must satisfy the Euler-Lagrange equation in (2.2) was carried over to (equation 2.3). In this case F is area and G is arc length (equation 2.3) is the standard isoperimetric problem. Historically, (equation 2.3) has been called a general isoperimetric problem, and the constraint $G(y) = l$ has been called a general isoperimetric constraint, even if it may not represent arc length. Euler derived the rule which we call (Euler's Multiplier Rule)which
"state that If y^* is a solution of the general isoperimetric problem (equation 2.2), then there exists an associated multiplier λ such that y^* is an extremal of the auxiliary problem"

Extremize L(y)= $f(Y) + \lambda G(Y)$

$$\text{subject to } y(a) = \alpha \text{and} y(b) = \beta \quad (2.4)$$

then y^* is an extremal of problem (2.3) if there exists an associated multiplier λ so that y^* is an extremal of problem (2.4) with this choice of λ
then the isoperimetric problem was considered in Queen Dido form

Maximize $\int_{-a}^{a} y(x)dx$

subject to $\int_{-a}^{a} \sqrt{1 + y'(x)^2}dx$

$$y(a) = y(a) = 0 \quad (2.5)$$

Euler observed that the semi-circle

$$y_c(x) = \sqrt{a^2 - x^2} - a \le x \le a \quad (2.6)$$

is an extremal of the Multiplier Rule auxiliary problem

Maximize $\int_{-a}^{a}(y(x)dx - \sqrt{1 + y'(x)^2})dx$

$$\text{Subject to} y(a) = y(a) = 0 \quad (2.7)$$

(equation 2.7) corresponds to a multiplier choice of $\lambda = -a$ in the Euler auxiliary problem for (equation 2.5). Notice that for problem (2.7),with f denoting the obvious quantity, we have

$$f_y = 1 and fy' = -a\frac{y'(x)}{\sqrt{1 + y'(x)^2}} \quad (2.8)$$

Now evaluating these quantities for equation (2.6) we see that

$$\frac{Y_C'(X)}{\sqrt{1+Y_C'(X)^2}} = \frac{x}{a} \tag{2.9}$$

Hence $f_{y'} = x$ and the semi-circle satisfies the Euler-Lagrange equation (2.2) for problem (2.7).
The name Lagrange was attached to the greatly innovative work of Euler. According to Goldstine in August 1755 a 19 year old Lagrange wrote Euler a brief letter to which was attached an appendix containing mathematical details of a beautiful and revolutionary idea, the notion of variations. This idea could be used to remove Euler's tedium and need for geometrical insight and all could be done with analysis using Lagrange's new notion of variations. Euler was so impressed that he dropped his own methods, espoused those of Lagrange, named the subject "the calculus of variations", and called multiplier theory Lagrange multiplier theory. Indeed today the Euler auxiliary functional $L(y)$ given in equation (2.4) is called the Lagrangian

In Queen Dido statement of the isoperimetric problem given by (2.5), it was observed that the semi-circle given by (2.6) has infinite slope at the end points $x = a$ and $x =-a$; so the arc length integral in problem (2.5) is not defined. Throughout the years, beginning with Euler and including contemporary times, authors have swept this undesirable aspect of the problem under the rug and just ignored it. Weierstrass dealt with the situation by stating the problem in parametric form. Many authors, for example Elsgolc consider the Euler-Lagrange equation for the Euler's auxiliary problem (2.7) as Euler did, but in order to solve the Euler-Lagrange equation they make a change of variables that leads to the solution in parametric form, and then they argue that elimination of the parameter gives the equation for a circle. Yet still other authors, for example Gelfand and Fomin consider the Euler-Lagrange equation as Euler did and then simply infer that integration of the equation leads to a family of circles.

Definition 2.1.1 (The Euler Class)
By E(-a, a) the Euler class of curves for the isoperimetric problem (2.5), we mean the collection of $y : [-a, a] \rightarrow \Re$ satisfying the following condition

1. $y(-a)] = y(a)$
2. y is continuous on [-a, a]
3. y is differentiable except possibly on a countable subset of [-a, a]

4. the arc length integral $\int_{-a}^{a} \sqrt{1 + y'(x)^2}dx$
5. Exists as a proper Riemann integral or
6. The curve y is rectifiable and the arc length integral exists as an improper Riemann integral.

2.2 A Proof Motivated by Euler.

Theorem 2.2.1

The semi-circle curve $y_c(x)$ given by (2.6) uniquely solves the isoperimetric problem (2.5), with the arc length integral interpreted as an improper Riemann integral, over E(-a, a), the Euler class of functions given by Definition (2.1.1)

Proof

Consider the objective function in Euler's auxiliary problem (2.7) for the isoperimetric problem of equation (2.5),

$$J(y) = \int_{-a}^{a} (y(x) - a\sqrt{1 + y'(x)^2})dx \tag{2.10}$$

and the semi-circle

$$y_c(x) = \sqrt{a^2 - x^2}, for - a \leq x \leq a \tag{2.11}$$

For the sake of convenience he considered the integral in $J(Y)$ as an improper integral, and nothing is lost if it exists as a proper integral. He also considered when $y \neq y_c$ is contained in the Euler class and let η denote $y - yc$. Then

$$\phi_t = \int_{-a+\epsilon}^{a-\epsilon} [y_c + t\eta - a\sqrt{1 + (y_c' + t\eta')^2}]dx \tag{2.12}$$

for $t\varepsilon[0, 1]$ and $\epsilon\varepsilon(0, a)$ Straightforward differentiations give

$$\phi_c'(t) = \int_{-a+c}^{a-c} [\eta - a\frac{(y'c + t\eta)\eta'}{\sqrt{1 + (y_c' + t\eta')}}]dx$$

$$\phi_\epsilon''(t) = -a\int_{a-\epsilon}^{-a+\epsilon} \frac{\eta'^2}{[1 + (y_c' + t\eta)^2]^{\frac{3}{2}}}dx$$

Taylor's Theorem tells us that

$$\phi_\epsilon'(1) = \phi_\epsilon(0) + \phi_\epsilon'(0) + \frac{1}{2}\phi_\epsilon''(\theta) for some \theta \varepsilon (0,1)$$

now we see that equation from definition (2.1.1) number (4) gives

$$\phi_\epsilon'(0) = \int_{-a+\epsilon}^{a-\epsilon} (\eta + x\eta) dx = x\eta(x) \mid_{-a+\epsilon}^{a-\epsilon}$$

and because η is continuous $\phi_\epsilon'(0) \to 0$ as $\epsilon \to 0$ Also observe that $\phi_\epsilon''(0) < 0$ and decreases as ϵ decreases. So letting $\epsilon \to 0$ now gives

$$J(Y) < J(y_c)$$

The fourth term in (2.7) must have a limit since the first three do. Now, restricting our attention to all y in the Euler class which have arc length equal to $a\pi$ (the arc length of the semi-circle) (2.8) tells us that y_c uniquely solves the isoperimetric problem in the Euler class.

Remark

It is important to realize that our sufficiency proof borrowed only the objective function J of the auxiliary problem (2.7) from Euler's necessity proof. Hence, it doesn't matter whether Euler's proof of his rule was rigorous or not.

2.3 A Proof Motivated by Lagrange

Theorem 2.3.1

The semi-circle curve $y_c(x)$ given by Equation(2.6) uniquely solves the iso-area problem
Minimize

$$J(Y) = \int_{-a}^{a} \sqrt{1 + y'(x)^2} dx$$

Subject to

$$\int_{-a}^{a} y(x) dx = \frac{\pi}{2} a^2$$

$$y(-a) = y(a) = 0$$

with the arc length integral interpreted as an improper Riemann integral, over E(-a' a), the Euler class of functions given by Definition (2.1.1) Hence it uniquely solves the isoperimetric problem, with the arc length integral interpreted as an improper Riemann integral, over the Euler class.

Proof

Following Lagrange's 1759 derivation of the Euler-Lagrange equation we first consider a class of admissible variations.

$$S = (\eta \epsilon E(-a,a) : \int_{-a}^{a} \eta(x)dx)$$

Since members of E(-a, a) are continuous the area integral in problem (2.8) and in (2.9) are viewed as proper Riemann integrals. As before, let y_c denote the semi-circle (2.6) and consider any $y \neq y_c$ contained in the Euler class. Let $\eta = y - y_c$ and notice that $\eta \epsilon \epsilon S$. As in the previous proof, define

$$\phi_\epsilon(t) = \int_{-a+\epsilon}^{a-\epsilon} \sqrt{1 + (y_c' + t\eta)^2} dx$$

for t $\varepsilon[0,1]$ and $\epsilon\varepsilon(0,a)$

Straightforward differentiations with respect to t give

$$\phi_\epsilon'(t) = \int_{-a+\epsilon}^{a-\epsilon} \frac{(y_c' + t\eta')\eta'}{\sqrt{1 + (y_c' + t\eta')^2}} dx$$

$$\phi_\epsilon''(t) = \int_{-a+\epsilon}^{a-\epsilon} \frac{(\eta')^2}{(1 + (y_c' + t\eta')^2)^{\frac{3}{2}} dx}$$

Recalling (2.9) and integration by parts give

$$\phi_\epsilon'(0) = -\frac{1}{a} \int_{-a+\epsilon}^{a-\epsilon} x\eta' dx = \frac{1}{a}[-x\eta(x) \mid_{-a+\epsilon}^{a-\epsilon} + \int_{-a+\epsilon}^{a-\epsilon} \eta dx]$$

Hence $\phi_\epsilon'(0) \to 0$ as $\epsilon \to 0$

Taylor's theorem tells us that

$$\phi_\epsilon(1) = \phi_\epsilon(0) + \phi_\epsilon'(0) + \tfrac{1}{2}\phi_\epsilon''(\theta) \text{ for some } \theta\varepsilon(0,1)$$

Observe that $\phi_\epsilon''(\theta) > 0$ and increases as ϵ decreases. So letting $\epsilon \to 0$ in the last equation gives

$$J(y_C) < Jy$$

Since y was an arbitrary member of the Euler class y_c uniquely solves the iso-area problem, hence the isoperimetric problem in the Euler class.

Remark

Lagrange could have made this proof because he was familiar with the form of Taylor's theorem that we used, while this hypothesized proof would have been made 50 years after Euler, it would still have been some 80 years before Weierstrass.

In conclusion, firstly is that the isoperimetric problem has been a most impactful mathematical problem. The isoperimetric problem, perhaps because it is so easy to state and understand and yet its solution has been so mathematically challenging has influenced the writings of scholars in many diverse areas.Euler built multiplier theory specifically to solve this problem, He worked with the iso-area problem and in a most ingenious manner made a coordinated transformation writing the curve under consideration in parametric form where the independent variable was arc length. When he wrote the transformed problem the area constraint vanished. Hence he arrived at a problem which had no subsidiary constraint and had the form of the simplest problem in the calculus of variations. He then showed that the circle was an extremal of this problem by solving the Euler-Lagrange equation associated with the transformed problem.

2.4 Different Use Of Lagrange Multipliers In Optimization And Economic

2.4.1 Optimization

Optimization problems which seek to minimize or maximize a real function play an important role in the real world, it can be classified into unconstrained optimization problems and constrained optimization problems. Our everyday life can be formulated as constrained optimization problems, for instance how to Maximize the profit of an investment is an example of a constrained optimization problem.

In unconstrained problems the stationary point gives the necessary condition to find the extrema point of the objective function $f(x_1, ..., x_n)$, the stationary points are the points where the gradient Δf is zero that is each of the Partial derivative is zero,
All the variables in $f(x_1, ..., x_n)$ are independent so they can be arbitrarily set to seek the extreme of f . However when it comes to the constrained optimization problems, the arbitrary of the variables does not exist. The constrained optimization problems can be formulated into a standard form as seen below

$$Min f(x_1, ..., x_n)$$

Subject to

$$G(x_1, ..., x_n) = 0$$

$$H(x_1, ..., x_n)$$

Where, G, H are functions, the variables are restricted to the feasible region which refers to the points satisfying the constraints. Substitution is an intuitive method to deal with in optimization problems, But these can be applicable to equality constrained optimization problems and often fails in most of the nonlinear constrained problems, where it is difficult to get the explicit expressions for the variables needed to be eliminated in the objective function.
The Lagrange multiplier method provides an alternative method for the constrained nonlinear optimization problems, it can help to deal with both equality and inequality constraints, e.g: given a function $f(x, y)$ subject to a constraint $g(x, y)$ a new function f can be formed, by applying lagrange multiplier thus we have

$$f(x, y, \lambda) = f(x, y) + \lambda(g(x, y))$$

Here $f(x, y, \lambda)$ is the lagrangian function, $f(x, y)$ is the objective function and $g(x, y)$ is the constraint, since the constraint is always set to be equal to zero the product $\lambda(g(x, y))$ is equal to zero, the addition of the term does not change the value of the objective function, the critical value x_0 y_0 and λ_0 at which the function is optimized are obtained by taking the partial derivatives of f with respect to all the three independent variable, setting them equal to zero and solving simultaneously

$$f_x(x, y, \lambda) = 0, f_y(x, y, \lambda) = 0, f_\lambda(x, y, \lambda) = 0$$

Example

Optimize the function $z = 4x^2 + 3xy + 6y^2$ subject to the constraint $x + y = 56$

In solving these we set the constraint equal to zero by subtracting the variable from the constant

$$56 - x - y = 0$$

applying the lagrange multiplier we have the lagrangian's function to be

$$z = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

taking the partial derivatives and setting them equal to zero and also solve simultaneously,we then have;

$$z_x = 8x + 3y - \lambda = 0 \quad (2.13)$$

$$z_y = 3x + 12y - \lambda = 0 \quad (2.14)$$

$$z_\lambda = 56 - x - y = 0 \quad (2.15)$$

subtracting equation 3.4 from 3.5 to we have

$$5x - 9y = 0, x = 1.8y$$

substituting $x = 1.8$y in equation 3.6 we have

$$56 - 1.8y - y = 0$$

$$y = 20$$

since $y = 20$,substituting y in equation 3.6 and x, y in any of the equation, we have

$$x = 36, and\lambda = 348$$

The method of lagrange multipliers can be extended to constrained optimization problems involving functions of more than two variables and more than one constraint.For instance, to optimize $f(x, y, z)$ subject to the constraint $g(x, y, z)$, we solve $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$. An example of a problem involving this kind of constrained optimization is below

Example

A jewelry box is to be constructed of material that costs 1 per square inch for the bottom, 2 per square inch for the sides, and 5 per square inch for the top. If the total volume is to be 96 in^3, what dimensions will minimize the total cost of construction?

Solution

Let the box be x inches deep y inches long, and z inches wide where x, y, and z are all positive, then the volume of the box is $V = xyz$ and the total cost of construction is given by

$$C = 1yz + 2(2xy + 2xz) + 5yz = 6yz + 4xy + 4xz$$

which are (bottom, side, and top) now we wish to minimize $C = 6yz + 4xy + 4xz$ subject to $V = xyz = 96$ .The lagrangian's function is now

$$6yz + 4xy + 4xz + \lambda(xyz)$$

now we find the partial derivatives of each variables and solve simultaneously

$$C_x = \lambda v_x or 4y + 4z = \lambda(yz)$$

$$C_y = \lambda v_y or 6z + 4x = \lambda(xz)$$

$$C_z \lambda v_z or 6y + 4x = \lambda(xy)$$

we now have three equations below

$$4y + 4z - \lambda yz = 0 \tag{2.16}$$

$$6z + 4x - \lambda xz = 0 \tag{2.17}$$

$$6y + 4x - \lambda xy = 0 \tag{2.18}$$

Solving each of the first three equations, we have

$$\frac{4y+4z}{yz} = \frac{6z+4x}{xz} = \frac{6y+4x}{xy} = \lambda$$

By multiplying each expression by xyz, we obtain

$$4xy + 4xz = 6yz + 4yx$$
$$4xy + 4xz = 6yz + 4xz$$
$$6yz + 4yx = 6yz + 4xz$$

which can be further simplified by common terms on both side of each equation to get

$$4xz = 6yz \tag{2.19}$$

$$4xy = 6yz \tag{2.20}$$

$$4yx = 4xz \tag{2.21}$$

By dividing z from equation(3.7),y from the equation(3.8),and x from equation(3.9),we obtain

$$4x = 6y \tag{2.22}$$
$$4x = 6z \tag{2.23}$$
$$4y = 4z \tag{2.24}$$

so that $y = \frac{2}{3}x$ and $z = \frac{2}{3}x$ and $y = z$, substituting these values into the constraint equation $xyz = 96$, we have

$$x(\frac{2}{3}x)(\frac{2}{3}x) = 96$$

$$\frac{4}{9}x^3 = 96$$

$$x^3 = 216 so x = 6$$

and then $y = z = \frac{2}{3}(6) = 4$
Thus, the minimal cost occurs when the jewelry box is 6 inches deep with a square base, 4 inches on a side.

2.4.2 The Power Systems Economic Operation

The Lagrange multiplier method is also used to solve extreme value problems in economics,in the cases where the objective function f and the constraints G have specific meanings, the Lagrange multipliers often has an identifiable significance.
In economics, if you're maximizing profit subject to a limited resource, λ is the resource's marginal value, sometimes called shadow price Specifically, the value of the Lagrange multiplier is the rate at which the optimal value of the objective function f changes if their is a change the constraints.
An important application of Lagrange multipliers method in power systems is the economic dispatch, or λ-dispatch problems.In this problem, the objective function to minimize is the generating cost, and the variables are subjected to the power balance constraint. This economic dispatch method is illustrated in the example below

Example

- Three generators with the following cost functions serve a load of 952Mw assuming a lossless system, calculate the optimal generation scheduling

$$f_1 : x_1 + 0.0625x_1^2$$

$$f_2 : x_2 + 0.0125x_2^2$$
$$f_3 : x_3 + 0.0250x_3^2$$

Where, x_i is the output power of the ith generator;f_i is the cost per hour of the generator. The cost function f_i with respect to x_i is generated by polynomial curve fitting based on the generator operation data, x_i has the unit MW. Since w = $\frac{j}{s}$ and the costs to produce 1j has the unit £, we have $[w] = [\frac{£}{s}] = [\frac{£}{hr}]$.Hence, the cost f_i has the unit $£/$hr.

The first step in determining the optimal scheduling of the generators is to express the problem in mathematical form. Thus the optimization statement is

$$Minf = f_1 + f_2 + f_3 = x_1 + 0.625x_1^2 + x_2 + 0.0125x_2^2 + x_3 + 0.0250x_3^2$$

Subject to

$$G = x_1 + x_2 + x_3 - 952 = 0$$

The corresponding Lagrangian's function is

$$f = x_1 + 0.625x_1^2 + x_2 + 0.0125x_2^2 + x_3 + 0.0250x_3^2 - \lambda(x_1 + x_2 + x_3 - 952)$$

Setting $\nabla f = 0$ yields the following set of linear equations:

$$\begin{pmatrix} 0.125 & 0 & 0 & -1 \\ o & 0.025 & 0 & -1 \\ 0 & 0 & 0.05 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 952 \end{pmatrix}$$

$$x_1 = 112Mw$$
$$x_2 = 560Mw$$
$$x_3 = 280Mw$$
$$\lambda = £15/Mwhr$$

and the constrained cost f is £7616/hr.

This is the generator scheduling that minimizes the hourly cost of production. The value of λ is the incremental break-even cost of production. This gives the company a price cut-off for buying or selling generator: if they can purchase generator for less than λ, then their overall costs will decrease. Likewise, if generator can be sold for greater than λ, their overall costs will decrease. Also note that at the optimal scheduling, the value of λ, and x_i satisfy

λ (£/Mwhr) $= 1 + 0.125x_1 = 1 + 0.025x_2 = 1 + 0.05x_3$ Since λ is the incremental cost for the system, this point is also called the point of equal incremental cost criterion. Any deviation in generator from the equal incremental cost scheduling will result in an increase in the production cost f.

Chapter 3

Methodology

The method of Lagrange multiplier allows us to maximize or minimize functions with specific conditions known as constraints. To find critical point of a function $f(x, y, z)$ subject to the constraint $g(x, y, z) =$ C we must solve the following system of simultaneous equations:

$$\nabla f(x, y, z) = \lambda g(x, y, z)$$
$$\lambda g(x, y, z) = C$$

Remembering that ∇f and ∇g are function, we can write this as a collection of four equations in the four unknowns $x, y, z,$ and λ

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$
$$f_y(x, y, z) = \lambda g_y(x, y, z)$$
$$f_z(x, y, z) = \lambda g_z(x, y, z)$$
$$g(x, y, z) = C$$

The variable λ is a dummy variable called a Lagrange multiplier"; we only care about the values of $x, y,$ and z .Once we have found all the critical points, we substitute them into f to determine the maxima and minima points.The critical points where f is greatest are maxima and the critical points where f is smallest are minima.

3.1 Procedure For Solving Lagrangian Equation

1. Since we don't actually care what λ is, we can first solve for λ in terms of $x, y,$ and z to remove λ from the equations.

2. we solving for one variable in terms of the others

3. In case of square root, we must consider both the positive and the negative square roots.

4. whenever we divide an equation by an expression, we must be sure that the expression is not 0, it may help to split the problem into two cases, first solve the equations assuming that a variable is 0, and then solve the equations assuming that it is not 0.

SIMPLE ILLUSTRATION OF PROCEDURE

Use Lagrange multipliers to find the maximum and minimum value of a function subjected to given constraints $x^2 + y^2 = 0$ if $f(x, y) = 3x + y$ For this kind of problem, $f(x, y) = 3x + y$ and $g(x, y) = x^2 + y^2 = 0$
Let us go through the steps (procedure)

$$\nabla f = (3, 1) \tag{3.1}$$

$$\nabla g = (2x, 2y) \tag{3.2}$$

$$(3, 1) = \lambda(2x, 2y) \tag{3.3}$$

we solve the above equation and consider the following system of 3 equations with 3 unknowns (x, y, λ)

$$2\lambda x = 3 \tag{3.4}$$

$$2\lambda y = 1 \tag{3.5}$$

$$x^2 + y^2 = 10 \tag{3.6}$$

Now solving for each variables,from equation (3.4) $x = \frac{3}{2\lambda}$ also from equation (3.5),$y = \frac{1}{2\lambda}$, Putting the value of x and y in equation (3.6) we have

$$(\frac{3}{2\lambda})^2 + (\frac{1}{2\lambda})^2 = 10$$

$$\lambda = \pm\frac{1}{2}$$

Now, we plug λ back into our original equation(3.1) and (3.2) $x = \pm\, 3$ and $y = \pm\, 1$ i.e. we have $(3, 1)$ and $(-3, -1)$.
We can classify them by simply finding their values and subtitue into $f(x, y)$ $= 3x + y$ then we have

1.

$$\nabla f(3, 1) = 9 + 1 = 10$$

2.

$$\nabla f(-3,-1) = -9 - 1 = -10$$

So the maximum happen at $(3,1)$ and the minimum happen at $(-3,-1)$, this is an example of an equality constraint.

3.2 Derivation Of Euler- Lagrange Equation

Given a function $y(x)$ which extremizes a given functional

$$I(y) = \int_{x_1}^{x_2} f(x,y,y')dx$$

Subject to the boundary condition

$$y(x_1) = y_1, y(x_2) = y_2,.$$

the solutions $y(x)$ of this problem are called the extremals of I (or of f), and the corresponding values of I are the extrema.In deriving the Euler-Lagrange equation, which provides a necessary condition for $y(x)$ to be an extremal of I. Suppose $y(x)$ is an extremal, i.e. a particular function which extremizes I, we consider small variations of the form

$$\tilde{y(x)} = y(x) + \varepsilon\eta(x)$$

where ε is a small parameter and $\eta(x)$ is any function satisfying the boundary conditions $\eta(x_1) = \eta(x_2) = 0$ (so that the variation $\tilde{y(x)}$ satisfies the same boundary conditions as the extremal $y(x)$. Since $y(x)$ is an extremal, the functional I should be stationary with respect to all such variations, i.e. we must have

$$\frac{dI}{d\varepsilon}|_{\varepsilon=0} = \lim_{\varepsilon\to 0}(\frac{I(\tilde{y}) - I(y)}{\varepsilon}) = 0$$

for every possible choice of $\eta(x)$. Let $\Delta I = I(\tilde{y}) - I(y)$. Then

$$\Delta I = \int_{x_1}^{x_2} f(x,\tilde{y},\tilde{y'})dx - \int_{x_1}^{x_2} f(x,y,y')dx = \int_{x_1}^{x_2} f(x,y+\varepsilon\eta,y'+\varepsilon\eta' dx) - \int_{x_1}^{x_2} f(x,y,y')dx$$

We expand the first integrand as a Taylor series, keeping only the leading terms:

$$\Delta I = \int_{x_1}^{x_2} (f(x,y,y') + \varepsilon\eta\frac{\partial F}{\partial y} + \varepsilon\eta'\frac{\partial F}{\partial y'} + 0(\varepsilon^2))dx - \int_{x_1}^{x_2} f(x,y,y')dx =$$

$$\varepsilon \int_{x_1}^{x_2} (\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'}) dx + 0(\varepsilon^2)$$

Now, integrating the second term by parts and applying the boundary conditions on η

$$\Delta I = \varepsilon [\eta \frac{\partial F}{\partial y}]_{x_1}^{x_2} + \varepsilon \int_{x_1}^{x_2} (\eta \frac{\partial F}{\partial y} - \eta \frac{d}{dx} (\frac{\partial F}{\partial Y'})) dx$$

Thus the requirement becomes

$$\lim_{\varepsilon \to 0} (\frac{\Delta I}{\varepsilon}) = \int_{x_1}^{x_2} \eta (\frac{\partial F}{\partial y} - \frac{d}{dx} (\frac{\partial F}{\partial y'})) dx = 0$$

and, because this must hold for every choice of $\eta(x)$, we deduce the Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} (\frac{\partial F}{\partial y'}) = 0 \tag{3.7}$$

3.3 Eulers Theorem

If a curve y = $y(x)$ extremizes the function $J[y] = \int_{x_0}^{x_1} f(x, y, y') dx$ subject to the condition $K[y] = \int_{x_0}^{x_1} G(x, y, y') dx = 1$, $y(x_0) = y_0$, $y(x_1) = y_1$, and if y = y(x)is not an extremal of the functional K, then there exists a constant λ such that the curve y = y(x)is an extremal of the functional

$$H = \int_{X_0}^{X_1} [F(x, y, y') + \lambda G(x, y, y')] dx$$

The vital condition for the solution of this problem is to satisfy the Euler-Lagrange equation (3.19)

$$\frac{\partial H}{\partial y} - \frac{d}{dx} (\frac{\partial H}{\partial y'}) = 0 \tag{3.8}$$

3.4 Other Forms Of Euler's Equation

1.

$$\frac{d}{dx} f(x, y, y') = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx}$$

or

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''$$

but

$$\frac{d}{dx}(y'\frac{\partial f}{\partial y'}) = y'\frac{d}{dx}(\frac{\partial f}{\partial y'}) + \frac{\partial f}{\partial y'}y''$$

on subtracting the second equation from the third equation we have

$$\frac{df}{dx} - \frac{d}{dx}(y'\frac{\partial f}{\partial y'}) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial}y' - y'\frac{d}{dx}(\frac{\partial f}{\partial y'})$$

$$\frac{d}{dx}[f - y'\frac{\partial f}{\partial y'}] - \frac{\partial f}{\partial x} = y'[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{df}{dy'}]$$

hence

$$\frac{d}{dx}[f - y'\frac{\partial f}{\partial y'}] - \frac{\partial f}{\partial x} = 0 \tag{3.9}$$

which is another form of Euler's equation

2. we know that $\frac{\partial f}{\partial y'}$ is also function x, y, y' say $\phi(x, y, y')$

$$\frac{d}{dx}(\frac{\partial f}{\partial y'}) = \frac{\partial \phi}{\partial x}\frac{dx}{dx} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} + \frac{\partial \phi}{\partial y'}\frac{dy'}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x}y' + \frac{\partial \phi}{\partial y'}y''$$

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y'}) + \frac{\partial}{\partial y}(\frac{\partial f}{\partial y'})y' + \frac{\partial}{\partial y'}(\frac{\partial f}{\partial y'})y'' = \frac{\partial^2 f}{\partial x \partial y'} + y'\frac{\partial^2 f}{\partial y \partial y'} + y''\frac{\partial^2 f}{\partial y'^2}$$

putting the value of $\frac{d}{dx}(\frac{\partial f}{\partial y'})$ in Euler's equation we have

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y'\frac{\partial f}{\partial x \partial y'} - y''\frac{\partial^2 f}{\partial y'^2} = 0 \tag{3.10}$$

This is the third form of Euler's equation

3.5 Extremal

Any function which satisfies Euler's Equation is known as Extremal is obtained solving the Euler's Equation

Case 1

If f is independent of x i.e. $\frac{\partial f}{\partial x} = 0$ in $\frac{d}{dx}[f - y'\frac{\partial f}{\partial y'}] - \frac{\partial f}{\partial x} = \mathbf{0}$ we have

$$\frac{d}{dx}[f - y'\frac{\partial f}{\partial y'}] = 0$$

Integrating we have $f - y'\frac{\partial f}{\partial y'} =$ constant.

Caes 2

when f is independent of y' i.e. $\frac{\partial f}{\partial y'} = \mathbf{0}$ putting the value of $\frac{\partial f}{\partial y}$ in Euler's equation we have $\frac{d}{dx}(\frac{\partial f}{\partial y'}) = \mathbf{0}$ integrating we have $\frac{\partial f}{\partial y'} = \mathbf{0}$ $=$ constant

Case 3

if f is an independent of y, i.e. $\frac{\partial f}{\partial y} = \mathbf{0}$ on substituting the value of $\frac{\partial f}{\partial y}$ in the Euler's equation we have $\frac{\partial f}{\partial y} = \mathbf{0}$ This is the desired solution .

Case 4

if f is independent of x and y we have that $\frac{\partial f}{\partial x} = \mathbf{0}$ and $\frac{\partial f}{\partial y} = \mathbf{0}$ or $\frac{\partial^2 f}{\partial x \partial y'} = \mathbf{0}$ and $\frac{\partial^2 f}{\partial y \partial y'} = \mathbf{0}$
putting these value in Euler's equation we have

$$y''\frac{\partial^2 f}{\partial y'^2} = 0$$

if $\frac{\partial^2 f}{\partial y'^2} \neq \mathbf{0}$ then $y'' = \mathbf{0}$ whose solution is $y = ax + b$

Chapter 4

Application Of Lagrange Multiplier To Calculus Of Variation

Calculus of variation is a field of mathematical analysis that deals with maximizing or minimizing functions, which are mapping from a set of functions to the real number. Functions are often expressed as definite integrals involving functions and their derivatives. The interest is in extremal function that makes the function attain a maximum or minimum value or stationary point when the rate of change of the function is zero. Calculus of variations primarily deals with finding maximum and minimum values of a definite integral involving a certain functional.
The calculus of variations originated with attempts to solve Queen Dido's problem, known to mathematicians as the Isoperimetric Problem to determine the shape of that closed plane curve of fixed length that encloses the maximum possible area of that plane. In ordinary functions the values of the independent variables are numbers. Whereas with functionals, the values of the independent variables are functions. Thus variation problems involve determination of maximum or minimum or stationary values of a functional. Consider these general functional

$$I(y(x)) = \int_{x_1}^{x_2} f(x, y, y')dx \tag{4.1}$$

A function $y = y_x$ which extremizes the equation above and satisfies the boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$ and is known as an extremal.

4.1 Standard Variational Problems(shortest distance)

Example 4.1

Find the shortest smooth plane curve joining two distinct points in the plane

Solution

Assume that the two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie in the xy-Plane. If $y = f(x)$ is the equation of any plane curve c in xy-Plane and passing through the points P_1 and P_2, then the length L of curve c is given by

$$L[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}$$

The variational problem is to find the plane curve whose length is shortest i.e., to determine the function $y(x)$ which minimizes the functional $L[y(x)]$. The condition for extrema is the Euler's equation (3.19)

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Here $f = \sqrt{1 + y'^2}$ so $\frac{\partial f}{\partial y} = \frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}}$
Then

$$0 - \frac{d}{dx}(\frac{y'}{\sqrt{1 + y'^2}}) = 0(\frac{y'}{\sqrt{1 + y'^2}}) = k,$$

where k is a constant

$$y' = k\sqrt{1 + y'}$$

squaring $y'^2 = k^2(1 + y'^2)$
i.e.$y' = \sqrt{\frac{k^2}{1+k^2} = M}$,and integrating $y = mx + c$, where c is the constant of integration.Thus the straight line joining the two points P_1 and P_2 is the curve with the shortest length (distance).

Example 4.2

find the stationary value of

$$I = \int_0^{\frac{\pi}{2}} [(\frac{dy}{dx})^2 + 2yx - y^2]dx$$

where **A**$(0,0)$**, and B** $(\frac{\pi}{2},\frac{\pi}{2})$ are the xy-plane,

Solution

Using Euler's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}$$

$$I=y'^2+2yx-y^2,\frac{\partial f}{\partial y}=2x-2yand\frac{\partial f}{\partial y'}=2y'$$

$$2x-2y-\frac{d}{dx}(2y')=0$$

$$2x-2y-2\frac{d^2y}{dx^2}=0$$

$$\frac{d^2y}{dx^2}+y=x$$

Our auxiliary equation becomes

$$m^2+1=0$$

$$m^2=-1$$

$$m=\sqrt{-1}$$

$$m=\pm i$$

$$y_c=A\cos x+B\sin x$$

$$y_p=EX+F$$

$$Y_P'=E$$

$$Y_P''=0$$

going back to our former equation,we have

$$0+EX+F=x$$

when F = 0, E = 1

$$y_p=x$$

general solution is now

$$y=y_c+y_p$$

$$y=x+A\cos x+B\sin x$$

where C and D are arbitrary constant.
However, in order that this curve passes through the two end-point $A(0,0), B(\frac{\pi}{2},\frac{\pi}{2})$ we have that At (0,0) = 0 + A + 0 = 0 i.e. A = 0
At $(\frac{\pi}{2},\frac{\pi}{2})$) B = 0

$$A = B = 0$$

consequently the function which makes stationary is

$$y = x$$

The stationary value of I,say by inserting the value of y into the integral and then integrating gives

$$I = \int_0^{\frac{\pi}{2}} (1 + 2x^2 - x^2)dx$$

$$x + \frac{2x^3}{3} - \frac{x^3}{3}$$

$$[x + \frac{x^3}{3}]_0^{\frac{\pi}{2}}$$

$$\frac{\pi}{2}(1 + \frac{\pi^2}{12})$$

Example 4.3

Consider the functional

$$J[y] = \int_1^2 \frac{\sqrt{1+y'^2}}{x} dxy(0) = 0, y(2) = 1$$

The integrand does not contain y,so we use the ideal in Case 2 in which the Eulers equation is of the form $Fy = C$, where C is a constant

$$f_y = \frac{y'}{x\sqrt{1+y'^2}}$$

$$\frac{y'}{x\sqrt{1+y'^2}} = C$$

so that

$$y'^2(1 - c^2x^2) = c^2x^2$$

Since C can take on either a positive or a negative value, we obtain the derivative from the above,

$$y'^2 = \frac{C^2x^2}{(1 - C^2x^2)}$$

$$y' = \frac{Cx}{\sqrt{1 - C^2x^2}}$$

Integrating the above gives us

$$y = \int \frac{Cx}{\sqrt{1 - C^2x^2}}$$

$$y = \frac{1}{C}\sqrt{1 - C^2x^2} + D$$

or equivalent to

$$y^2 + x^2 = \frac{1}{C^2}$$

The solution is a circle centered on the y-axis,From the boundary conditions at y(2) = 1 we have

$$D^2 - 2D + 5 = \frac{1}{C^2}$$

At y(1) = 0

$$D^2 + 1 = \frac{1}{C^2}$$

we now have two equations,Solving the two equations simultaneously,we have

$$C = \frac{1}{\sqrt{5}}, and D = 2$$

Thus, the final solution is

$$(y - 2)^2 + x^2 = 5$$

Example 4.4

Find the stationary value of

$$I = \int (\frac{dy}{dx})^2 + 2yx - y^2 where A(0,0) and B(\frac{\pi}{2}, \frac{\pi}{2})$$

Solution

The functional

$$f(x, y, y') = y'^2 + 2yx - y^2$$

Using Euler's equation

$$\frac{d}{dx}(\frac{\partial f}{\partial y'}) - \frac{\partial f}{\partial y}$$

therefore

$$\frac{\partial f}{\partial y'} = 2y'\frac{\partial f}{\partial y} = 2x - 2y$$

$$\frac{d}{dx}(2y') - (2x - 2y) = 0$$

$$\frac{d}{dx}(2y') - 2x + 2y = 0$$

$$2y'' - 2x + 2y = 0$$

$$y'' - x + y = 0$$

$$m^2 + 1 = 0$$

$$m = \pm i$$

since **m** = ± i the complimentary solution is given as

$$y_c = A\cos x + B\sin x$$

and the particular solution is

$$y_p = Ex + f$$

$$y' = E$$

$$Y'' = 0$$

So the general solution becomes

$$0 + Ex + f = x$$

$$E = 1, F = 0$$

$$y_p = x$$

$$y = y_c + y_p$$

$$y = A\cos x + B\sin x + x$$

from our boundary condition,we were given (0, 0) and $(\frac{\pi}{2}, \frac{\pi}{2})$ at y = 0, and x = 0 we have, C = 0 and at y = $\frac{\pi}{2}$, and x =$\frac{\pi}{2}$ so D = 0 So the general solution is y = x hence the equation becomes

$$\int_A^B (\frac{\partial y}{\partial x})^2 + 2yx + y^2 dx$$

$$= \int_0^{\frac{\pi}{2}} (1 + 2x^2 - x^2)dx$$

$$= \int_0^{\frac{\pi}{2}} (1+x^2)dx$$

$$(x+\frac{X^3}{3})|_0^{\frac{\pi}{2}}$$

$$\frac{\pi}{2}+\frac{\pi^3}{24}=0$$

$$\frac{\pi}{2}(1+\frac{\pi^2}{12})$$

4.2 Isoperimetric Problems

In a simple seance isoperimetric problems involves the determination of the shape of a closed curve of the given perimeter enclosing maximum area (the so-called Dido's problem). The determination of the shape of a closed curve of the given perimeter enclosed maximum area is an example of isoperimetric problem. The solution to this kind of problems is always a circle. Isoperimetric (iso for same, per metric for perimeter) problems deal with finding the closed curve of given length with an enclose maximum area subject to the constraint (condition) in which the length of the curve is fixed.

The simplest isoperimetric problem consists of finding a function $f(x)$ which extremizes the functional as shown

$$I = \int_{x_1}^{x_2} f(x,y,y')dx$$

subject to the constraint (condition)

$$J = \int_{x_1}^{x_2} g(x,y,y')dx$$

Assuming we are given prescribed values that satisfies the prescribed end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$,to solve this problem, we use the method of Lagrange multiplier,we now have a new form of function called the lagrangian.

$$H(x,y,y') = f(x,y,y') + \lambda g(x,y,y')$$

Where λ is an arbitrary constant known as the Lagrange multiplier. Now the problem is to find the extremal of the new functional.

$$\int_{x_1}^{x_2} H(x,y,y')dx$$

subject to no constraints (except the boundary conditions). Then the modified Eulers equation is given by

$$\frac{\partial H}{\partial y} - \frac{d}{dx}(\frac{\partial H}{\partial y'}) = 0$$

Examples 4.4

1. Find the curve C having length L which encloses a maximum area. The area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy + ydx$

Solution

Our curve has a length

$$ds^2 = dx^2 + dy^2$$

$$ds^2 = dx^2[1 + (\frac{dy}{dx})^2]$$

$$ds = \sqrt{(1 + y'^2)}dx$$

$$s = \int \sqrt{(1 + y'^2)}dx$$

the same as the length, which is L we now have to

maximize area

$$\frac{1}{2}\oint xdy + ydx$$

subject to length

$$\int \sqrt{(1 + y'^2)}dx = L$$

Applying lagrange multiplier we reduce the problem to an unconstrained one,so that we now have

Max
$$\frac{1}{2}\oint xdy + ydx + \lambda\int\sqrt(1+y'^2)dx$$

where
$$M = \frac{xy'-y}{2} + \lambda(1+y'^2)^{\frac{1}{2}}$$

This is were the Euler's Equation comes in,we now have our Euler's Equation to be

$$\frac{d}{dx}(\frac{\partial M}{\partial y'}) - \frac{\partial M}{\partial y} = 0$$

$$\frac{\partial M}{\partial y'} = \frac{x}{2} + \lambda y'(1+y'^2)^{-\frac{1}{2}}\frac{\partial M}{\partial y} = -\frac{1}{2}$$

$$\frac{d}{dx}(\frac{\partial M}{\partial y'}) = \frac{1}{2} + \lambda\frac{d}{dx}[y'(1+y'^2)^{-\frac{1}{2}}]$$

$$\frac{d}{dx}(\frac{\partial M}{\partial y'}) - \frac{\partial M}{\partial y} = \frac{1}{2} + \lambda\frac{d}{dx}[y'(1+y'^2)^{-\frac{1}{2}}] + \frac{1}{2} = 0$$

$$\lambda\frac{d}{dx}[y'(1+y'^2)^{-\frac{1}{2}}] = -1$$

$$\int d[y'(1+y'^2)^{-\frac{1}{2}}] = -\int dx$$

$$\frac{\lambda y'}{(1+y'^2)^{\frac{1}{2}}} = -x + k$$

$$\lambda y' = (-x+k)(1+y'^2)^{\frac{1}{2}}$$

$$\lambda^2 y'^2 = (-x+k)^2 + (1+y'^2)$$

$$(k-x)^2 + (k-x)^2 y'^2$$

$$y'^2[\lambda^2 - (k-x)^2] = (k-x)^2$$

$$y'^2 = \frac{(k-x)^2}{\lambda - (k-x)^2}$$

$$(\frac{dy}{dx})^2 = \frac{(k-x)^2}{\lambda - (k-x)^2}$$

$$\frac{dy}{dx} = \frac{(k-x)}{\sqrt{\lambda^2 - (k-x)^2}}$$

$$y = \int \frac{k-x}{\sqrt{\lambda^2 - (k-x)^2}} dx + B$$

now integrating using substitution method we have that

$$y = \frac{1}{2}.2u\frac{1}{2} + C$$

$$y = \sqrt{\lambda^2 - (k-x)^2} + C$$
$$Y - C = \sqrt{\lambda^2 - (k-x)^2}$$
$$(Y-C)^2 + (x-k)^2 = \lambda^2$$

The maximum area of the isoperimetric problem is a circle center at (k, c)

2. Find the solid of volume by the revolution of a given surface area such that the curve passes through the origin and it rotate about the x-axis,given a functional V = $\int_0^a \pi y^2$ subject to S = $\int_0^a 2\pi y\sqrt{(1+y^2)}dx$

solution

Here we have to extremize V with a given functional S, $V = \pi Y^2$ and $S = 2\pi y\sqrt{(1+y^2)}$ then

$$f = \pi y^2 + 2\pi y\sqrt{(1+y^2)}$$

for maximizing V,f must satisfy the Euler's equation. But v does not contain x so we use the other form of Euler's equation

$$f - y'\frac{dy}{dy'} = C$$

so that

$$\pi y^2 + \lambda 2\pi y\sqrt{(1+y'^2)} - y'\frac{1}{2}\frac{2\pi y\lambda 2y'}{\sqrt{(1+y'^2)}} = C$$

$$\pi y^2 + 2\pi Y\lambda\sqrt{(1+y'^2)} - \frac{2\pi\lambda yy'}{\sqrt{(1+y'^2)}} = C$$

$$\pi y^2 + \frac{2\pi y\lambda}{\sqrt{(1+y'^2)}} = C$$

As the curve passes through the origin (0,0) C = 0

$$\pi y^2 + \frac{2\pi y\lambda}{\sqrt{(1+y'^2)}} = 0$$

$$y + \frac{2\lambda}{\sqrt{(1+y'^2)}} = 0$$

$$y\sqrt{(1+y'^2)} = -2\lambda$$

$$1 + y'^2 = \frac{4\lambda^2}{y^2} \Rightarrow y'2 = \frac{4\lambda^2}{y^2} - 1 = \frac{4\lambda^2 - y^2}{y^2}$$

then

$$y' = \sqrt{\frac{4\lambda^2 - y^2}{y^2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{4\lambda^2 - y^2}}{y}$$

$$\int \frac{ydy}{\sqrt{4\lambda^2 - y^2}} = \int dx + C$$

$$-\sqrt{4\lambda^2 - y^2} = x + C$$

$$\sqrt{4\lambda^2 - y^2} = -x - C$$

$$(4\lambda - y^2) = (-x - C)^2$$

$$4\lambda^2 = (-x - C)^2 + Y^2$$

$$\lambda^2 = \frac{(-x - C)^2}{4} + \frac{y^2}{4}$$

3. Maximize

$$J(y) = \int_{-a}^{a} ydx$$

subject to the constraints y(-a) = y(a) = 0 and

$$K(y) = \int_{a}^{a} \sqrt{1 + y'^2} dx = l$$

Solution

Applying largrange multiplier and using Eulers Rule,since $J = y$, $K = \sqrt{1 + y'^2}$ we have $J(Y) + \lambda K(y)$

$$H = F + \lambda G = y + \sqrt{1 + y'^2}$$

$$H(y) = \int_a^a y + \sqrt{1 + y'^2}$$

Make $H(y)$ stationary and hence solve using Euler's equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx}(\frac{\partial H}{\partial y'}) = 0$$

$$1 - \frac{d}{dx}(\frac{\lambda y'}{\sqrt{1 + y'^2}}) = 0$$

$$\frac{d}{dx}(\frac{\lambda y'}{\sqrt{1 + y'^2}}) = 1$$

integrating both side we have

$$\frac{\lambda y'}{\sqrt{1 + y'^2}} = x + C$$

$$\frac{(\lambda y')^2}{1 + y'^2} = (x + C)^2$$

$$\lambda^2 Y'^2 = (x + C)^2 (1 + y'^2)$$

$$\lambda^2 y'^2 = (x^2 + 2xC + c^2)(1 + y'^2)$$

$$x^2 + 2xC + c^2 + x^2 y'^2 + 2xCy' + C^2 y'^2$$

$$(x + C)^2 + (x + C)^2 y'^2$$

$$\lambda^2 y'^2 - (x + C)^2 y'^2 = (x + C)^2$$

$$Y'^2(\lambda^2 - (X + c)^2) = (x + C)^2$$

$$y'^2 = \frac{(x+c)^2}{\lambda^2 + (x+C)^2}$$

$$y' = \frac{(x+C)}{\sqrt{\lambda^2 - (x+C)^2}}$$

Now by integrating both side we have

$$y = -(\lambda^2 - (x+C)^2)^{\frac{1}{2}} + C_1$$

$$(Y - C)^2 = \lambda^2 - (x+C)^2$$

The extrema of $J(y)$ is a circle,centred at (c, c_1) and radius $r = \lambda\lambda^2$ or $H = y + \sqrt{1 + y'^2}$.However, note that $\frac{\partial H}{\partial X} = 0$ which is the first integral

$$H - y' L_{y'} = constant$$

The equation above can be written as

$$(x + C_1)^2 + (Y - C_2)^2 = \lambda^2$$

This must satisfy the conditions that $y(a) = 0 = y(a)$ and $K(y) = L$

Chapter 5

Conclusion And Recommendation

Conclusion

The calculus of variations is concerned with finding the maxima and minima of certain functions,functional minimization problems known as variational problems which appear in various fields of science. It is also applicable to functions subject to more than one constraint.

In isoperimetric problems the idea behind lagrange multiplier method is to reduce constrained optimization problems to unconstrained optimization problems and solved using Euler's equation. Generally, Lagrange multiplier is very useful in the area where we are to get the exterma of a function (maximum and minimum) under a given condition known as constraint.

With the use of Lagrange multiplier in calculus of variation, it has made it easier in the field to arrive at the maxima and minima (extrema) of a function, in isoperimetric problems even when the constraint is an integral constraint.Lagrange multiplier has played a vital role in many other fields of science, social science and engineering,therefore Lagrange multiplier is very fast and accurate.

Recommendation

The method of Lagrange multiplier is very accurate and efficient for solving constraint problems in various fields, problems with two or more constraint. Differential transformation (numerical method) can also be used to solve problems in calculus of variation (isoperimetric problems). Therefore a software should be developed for Lagrange with the interest that it will be free from human error or error computation to make it easier and more time friendly in case of two or more constraint.

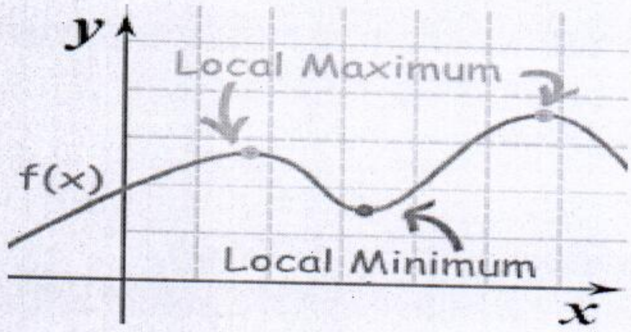

Figure 5.1: 1

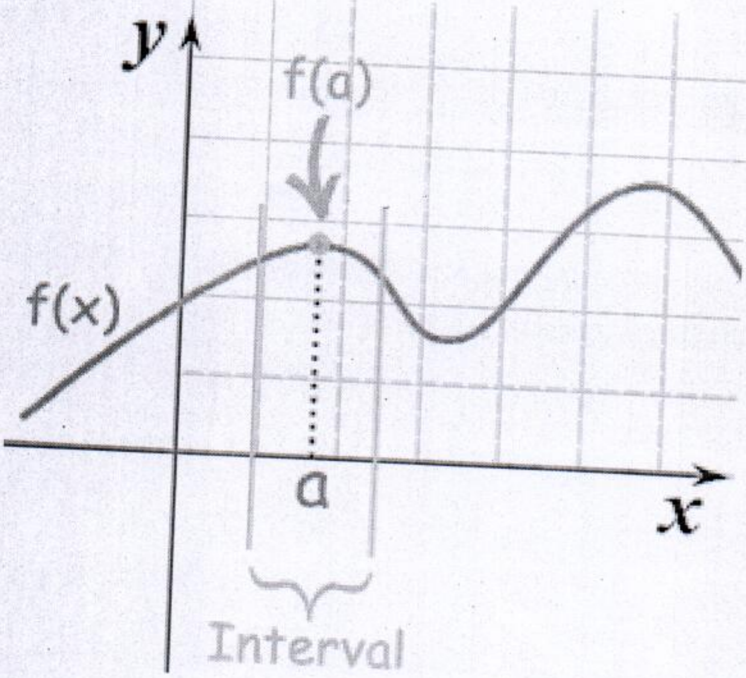

Figure 5.2: 2

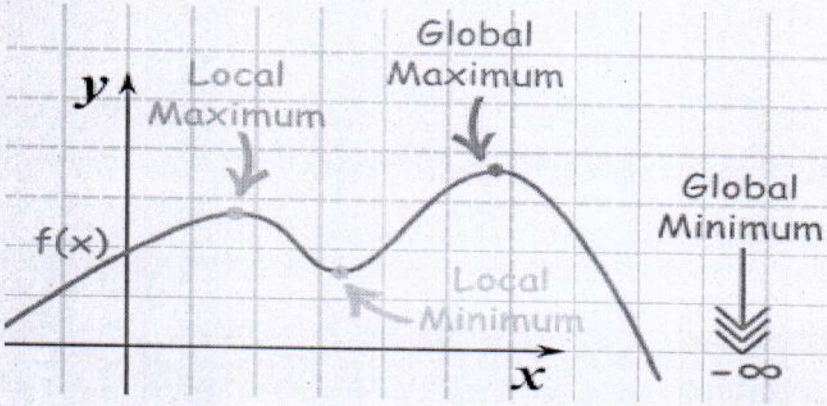
y
Local
Maximum
Global
Maximum
Global
Minimum
f(x)
Local
Minimum
x
-∞

Figure 5.3: 3